

UNIVERSITY OF CALIFORNIA, SANTA BARBARA
Department of Physics

Physics 110A
Prof. M.S. Witherell

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FINAL

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SHOW YOUR WORK AS CLEARLY AS POSSIBLE

The exam is closed book and notes, but you may use any information you have brought with you on one side of a $8\frac{1}{2}'' \times 11''$ sheet of paper. Do not spend too much time on one problem. Be sure to give answers in terms of given quantities.

#	pts
1	15
2	15
3	11
4	12
5	14
6	14
7	15
<hr/>	
	96

~~100~~

Good Work!

COURSE A+

1. A sphere of radius a contains charge of constant density:

$$\begin{aligned} \rho &= \rho_0 & r < a \\ \rho &= 0 & r > a \end{aligned}$$

a) Find ϕ everywhere.

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b) Calculate the electrical energy stored, using $U_e = \frac{1}{2} \int \rho \phi d\tau$.

c) Calculate the electrical energy stored, using $U_e = \frac{1}{2} \int \epsilon_0 E^2 d\tau$.



a) $r > a$

$$\phi = \frac{Q_{in}}{4\pi\epsilon_0 r} = \frac{\rho_0 a^3}{3\epsilon_0 r}$$

$r < a$

$$\vec{E} = \frac{Q_{in}}{4\pi\epsilon_0 r^2} \quad Q_{in} = \frac{4}{3}\pi r^3 \rho_0 \quad \vec{E} = \frac{\rho_0 r}{3\epsilon_0} \quad \phi(a) = \frac{\rho_0 a^2}{3\epsilon_0}$$

$$\phi(r) = \int_r^a \vec{E} \cdot d\vec{r} + \phi(a) = \frac{\rho_0}{3\epsilon_0} \int_r^a r dr = \frac{\rho_0}{6\epsilon_0} r^2 \Big|_r^a = \frac{\rho_0}{6\epsilon_0} (a^2 - r^2)$$

$$\Rightarrow \phi(r) = \frac{\rho_0}{6\epsilon_0} (a^2 - r^2) + \frac{\rho_0 a^2}{3\epsilon_0}$$

$$\frac{1}{3} - \frac{1}{5} = \frac{5}{15} - \frac{3}{15} = \frac{2}{15}$$

b) $U_e = \frac{1}{2} \int \rho \phi d\tau = \frac{1}{2} \rho_0 \int_0^a \int_0^{2\pi} \int_0^\pi \left(\frac{\rho_0}{6\epsilon_0} (a^2 - r^2) + \frac{\rho_0 a^2}{3\epsilon_0} \right) r^2 dr d\theta d\phi$

$$= \frac{4\pi\rho_0^2}{2 \cdot 6\epsilon_0} \int_0^a (a^2 - r^2) r^2 dr + \frac{4\pi\rho_0^2 a^2}{6\epsilon_0} \int_0^a r^2 dr = \frac{4\pi\rho_0^2}{12\epsilon_0} \left[\frac{a^2 r^3}{3} \Big|_0^a - \frac{r^5}{5} \Big|_0^a \right] + \frac{4\pi\rho_0^2 a^2}{6\epsilon_0} \left[\frac{r^3}{3} \Big|_0^a \right]$$

$$= \frac{4\pi\rho_0^2}{12\epsilon_0} \left[\frac{a^5}{3} - \frac{a^5}{5} \right] + \frac{4\pi\rho_0^2 a^5}{6\epsilon_0 \cdot 3} = \frac{\rho_0^2 a^5 \pi}{\epsilon_0} \left(\frac{8}{12 \cdot 15} + \frac{4}{18 \cdot 9 \cdot 5} \right) = \frac{12 \rho_0^2 a^5 \pi}{45 \epsilon_0}$$

$$\frac{12}{45}$$

Work Space

$$5 \text{ c) } U_e = \frac{1}{2} \epsilon_0 \int_V E^2 d\tau$$

$$r > a \quad \vec{E} = \frac{\rho_0 a^3}{3\epsilon_0 r^2}$$

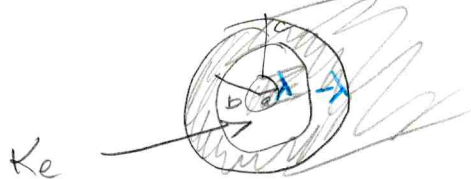
$$r < a \quad \vec{E} = \frac{\rho_0 r}{3\epsilon_0}$$

$$= \frac{1}{2} \epsilon_0 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_0^a \left(\frac{\rho_0 r}{3\epsilon_0}\right)^2 r^2 dr d\cos\theta d\phi + \frac{1}{2} \epsilon_0 \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \int_a^{\infty} \left(\frac{\rho_0 a^3}{3\epsilon_0 r^2}\right)^2 r^2 dr d\cos\theta d\phi$$

$$= \frac{2\pi \rho_0^2}{9\epsilon_0} \int_0^a r^4 dr + \frac{2\pi \rho_0^2 a^6}{9\epsilon_0} \int_a^{\infty} \frac{1}{r^2} dr \quad r^2 \cdot \left. \frac{-1}{r} \right|_a^{\infty} = \frac{1}{a}$$

$$= \frac{2\pi \rho_0^2 a^5}{45\epsilon_0} + \frac{5 \cdot 2\pi \rho_0^2 a^5}{5 \cdot 9\epsilon_0} = \boxed{\frac{12}{45} \frac{\rho_0^2 a^5 \pi}{\epsilon_0}}$$

Kelly



2. A coaxial cable consists of two conductors, one with outer radius a and the second with inner radius $b > a$, outer radius c . The gap between the conductors ($a < \rho < b$) is filled with a dielectric with dielectric constant κ_e . The inner conductor carries linear charge density λ , the outer carries density $-\lambda$.

- a) What is \vec{E} between the conductors?
- b) What is the capacitance per unit length?
- c) What is the volume bound charge density in the dielectric (ρ_b)?
- d) What is the surface bound charge density, σ_b , on the outer surface of the dielectric?
- e) What is the energy stored per unit length?

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3 a) $\vec{D} = \frac{\lambda}{2\pi\rho} \hat{\rho} \Rightarrow \vec{E} = \frac{\vec{D}}{\kappa_e \epsilon_0} = \boxed{\frac{\lambda}{2\pi\epsilon_0 \rho \kappa_e} \hat{\rho}}$ ✓

3 b) $C = \frac{Q}{\Delta\phi}$ $\Delta\phi = + \int_a^b E_{\rho} d\rho = + \frac{\lambda}{2\pi\epsilon_0 \kappa_e} \int_a^b \frac{d\rho}{\rho} = + \frac{\lambda}{2\pi\epsilon_0 \kappa_e} \ln \frac{b}{a}$
 $Q = \lambda L$

$C = \frac{\lambda L 2\pi\epsilon_0 \kappa_e}{\lambda \ln(b/a)} = \boxed{\frac{2\pi\epsilon_0 \kappa_e}{\ln(b/a)}}$ ✓

3 c) $\vec{P} = (\kappa_e - 1)\epsilon_0 \vec{E} = \frac{(\kappa_e - 1)\lambda}{\kappa_e 2\pi\rho} \hat{\rho}$

$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho P_{\rho}) = \boxed{0}$ ✓

3 d) $\sigma_b = \vec{P} \cdot \hat{n} = \boxed{\frac{(\kappa_e - 1)\lambda}{\kappa_e 2\pi b}}$ ✓ inner sum 0?

3 e) $U_e = \frac{1}{2} \int_V \vec{D} \cdot \vec{E} d\vec{r}$ $\vec{D} = \frac{\lambda}{2\pi\rho} \hat{\rho}$ $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 \rho \kappa_e} \hat{\rho}$
 $= \frac{\lambda^2}{8\pi^2 \epsilon_0 \kappa_e} \int_0^L \int_0^{2\pi} \int_a^b \frac{1}{\rho^2} \rho d\rho d\phi dz = \frac{L\lambda^2}{4\pi\epsilon_0 \kappa_e} \ln \rho \Big|_a^b = \boxed{\frac{\lambda^2 \ln(b/a)}{4\pi\epsilon_0 \kappa_e}}$ ✓

